Supplementarity measures on fuzzy sets*

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Abstract

In this paper, we commence the study of the so-called supplementarity measures. They are introduced axiomatically and are then related to incompatibility measures by antonyms. To do this, we have to establish what we mean by antonymous measure. We then prove that, under certain conditions, supplementarity and incompatibility measures are antonymous. Besides, with the aim of constructing antonymous measures, we introduce the concept of involution on the set made up of all the ordered pairs of fuzzy sets. Finally, we obtain some antonymous supplementarity measures from incompatibility measures by means of involutions.

Keywords: Antonymity in fuzzy sets, supplementary fuzzy sets, antonymous measures, supplementarity measures

1. Introduction

Variable labelling is a preliminary step in the design of fuzzy inference systems by an expert [14]. This labelling is generally far from straightforward, and system efficiency largely depends on correct label design.

A condition governing the choice of variable labels is that they should consistently cover all the possible characteristics of the phenomenon under examination, that is, somehow assure "supplementarity". In this paper, we set out to study this property for the limited case of two labels, leaving the modelling of a higher number for a later stage of our research. Research into other similar concepts, such as contradictory sets [4] and incompatible sets [2], could come in useful for this study.

Let us recall the classical case: Two subsets A and B of a universal set X are said to be contradictory if $A \subset B'$, where B' denotes the complement of B in X. Two sets A and B are incompatible if $A \cap B = \emptyset$, that is, they have no element in common. Finally, two subsets A and B of a universal set X are said to be supplementary if $A \cup B = X$, that is, if their union is the whole set.

Moving on to the fuzzy case: Let two fuzzy sets of a universe X with membership functions μ and σ, we say that:

1. They are N-contradictory, where N is a strong negation, if for all $x \in X$, we have that $\mu(x) \leq N(\sigma(x))$.
2. They are T-incompatible, where T is a t-norm, if for all $x \in X$ we have that $T(\mu(x), \sigma(x)) = 0$.
3. Finally, they are S-supplementary, where S is a t-conorm, if for all $x \in X$ we have that $S(\mu(x), \sigma(x)) = 1$.

The paper is organized as follows. Section 2 focuses on the definition of antonymous measures, supplementarity measures, and some early results about the relationship between supplementarity measures and incompatibility measures. Then, Section 3 deals with the construction of supplementarity measures from the incompatibility measures using antonyms. Finally, we present some conclusions.

2. Supplementarity measures with two arguments

As supplementarity can be considered to be an imprecise concept, it makes sense to examine this notion in the fuzzy sets domain.

We denote by $[0,1]^X$ the set of all the membership functions of fuzzy sets, and, for simplicity's sake, we often refer to $\mu \in [0,1]^X$ directly as a fuzzy set. For our purposes, remember that $([0,1]^X, \leq)$, where $\leq$ is the partial order obtained directly from the usual order on $\mathbb{R}$, is a bounded and complete lattice in which the least and the greatest elements are, respectively, $\mu_\emptyset, \mu_X \in [0,1]$ defined by $\mu_\emptyset(x) = 0$ and $\mu_X(x) = 1$ for all $x \in X$. As our objective is to model how to assign values to each pair of fuzzy sets, we first establish a suitable structure in the set of ordered pair of fuzzy sets, that is, $[0,1]^X \times [01]^X$. We can then tackle the formal model for measuring the supplementarity between two fuzzy sets.

2.1. Monotonic measures with two fuzzy sets as arguments

Let $\mathbb{I} = [0,1]^2$ be the unit square and consider the binary relation $\leq_{\mathbb{I}}$ on $\mathbb{I}$ defined as follows. Given $\bar{a} = (a_1, a_2), \bar{b} = (b_1, b_2) \in \mathbb{I}$,

$$\bar{a} \leq_{\mathbb{I}} \bar{b} \iff a_1 \leq b_1 \quad \text{and} \quad a_2 \leq b_2.$$

$(\mathbb{I}, \leq_{\mathbb{I}})$ is a partially ordered, bounded and complete lattice whose least element is $0_{\mathbb{I}} = (0,0)$ and whose greatest element is $1_{\mathbb{I}} = (1,1)$.

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Given $X \neq \emptyset$, the order $\leq_{\mathbb{I}}$ of $\mathbb{I}$ naturally induces a partial order on $\mathbb{I}^X = \{\bar{\mu} = (\mu_1, \mu_2) : X \to \mathbb{I}\} = [0,1]^X \times [0,1]^X$ as follows. Given $\bar{\mu} = (\mu_1, \mu_2), \bar{\sigma} = (\sigma_1, \sigma_2) \in \mathbb{I}^X$,

$$\bar{\mu} \leq_{\mathbb{I}} \bar{\sigma} \iff \left\{ \begin{array}{l} \mu_1(x) \leq \sigma_1(x) \\ \mu_2(x) \leq \sigma_2(x) \end{array} \right. \quad \forall x \in X.$$

Thus $(\mathbb{I}^X, \leq_{\mathbb{I}})$ is also a bounded and complete lattice in which the least and the greatest elements are, respectively, the functions $\bar{\mu}_\emptyset$ and $\bar{\mu}_X$, defined by $\bar{\mu}_\emptyset(x) = 0_{\mathbb{I}}$ and $\bar{\mu}_X(x) = 1_{\mathbb{I}}$ for all $x \in X$.

Definition 2.1. Let $X \neq \emptyset$, given $M : \mathbb{I}^X \to [0,1]$:

1. M is *a monotonic increasing function* if $M(\bar{\mu}) \leq M(\bar{\sigma})$ holds for all $\bar{\mu}, \bar{\sigma} \in \mathbb{I}^X$ such that $\bar{\mu} \leq_{\mathbb{I}} \bar{\sigma}$.
2. M is *a monotonic decreasing function* if $M(\bar{\mu}) \geq M(\bar{\sigma})$ holds for all $\bar{\mu}, \bar{\sigma} \in \mathbb{I}^X$ such that $\bar{\mu} \leq_{\mathbb{I}} \bar{\sigma}$.

We say that M is $\leq_{\mathbb{I}}$*-monotonic* if it satisfies either of these properties.

Note that $M : \mathbb{I}^X \to [0,1]$ is a function whose arguments are two fuzzy sets of X. Moreover, M is actually a membership function of a fuzzy set (or, simply, a fuzzy set) on $\mathbb{I}^X = [0,1]^X \times [0,1]^X$.

Remark 2.2. Let $M : \mathbb{I}^X \to [0,1]$ with $X \neq \emptyset$. If M is monotonic increasing, $M(\bar{\mu}_\emptyset) = 0$ and $M(\bar{\mu}_X) = 1$, then, according to Trillas and Alsina [10], M is a fuzzy $\leq_{\mathbb{I}}$-measure on $\mathbb{I}^X$. In the same way, with the reverse order of $\leq_{\mathbb{I}}$ in $\mathbb{I}^X$ ($\bar{\mu} \geq_{\mathbb{I}} \bar{\sigma} \Leftrightarrow \bar{\mu} \leq_{\mathbb{I}} \bar{\sigma}$), if M is monotonic decreasing, $M(\bar{\mu}_\emptyset) = 1$ and $M(\bar{\mu}_X) = 0$, then M is a fuzzy $\geq_{\mathbb{I}}$-measure on $\mathbb{I}^X$.

We denote by $\mathbb{M}^2([0,1]^X)$ the set of all functions whose arguments are two fuzzy sets of X, $M : \mathbb{I}^X \to [0,1]$, which are fuzzy $\leq_{\mathbb{I}}$-measures or $\geq_{\mathbb{I}}$-measures. If $M \in \mathbb{M}^2([0,1]^X)$, we say that M is a $\leq_{\mathbb{I}}$-monotonic fuzzy measure.

2.2. Defining supplementarity measures

Now, we will try to determine to what extent the union of two fuzzy sets covers the universe (i.e. they are supplementary). To do this, we need to consider a t-conorm S to model the union of sets. Thus, given the membership functions of two fuzzy sets, $\mu_1, \mu_2 \in [0,1]^X$, we want to establish to what extent $S(\mu_1, \mu_2) = \mu_X$ is satisfied. Remember that a function $S : \mathbb{I} \to [0,1]$ is a t-conorm [1, 6, 7, 8] if it is commutative, associative, monotonic increasing and satisfies $S(a, 0) = a$ for all $a \in [0,1]$. The t-conorm S naturally induces the union of fuzzy sets as a function $S : \mathbb{I}^X \to [0,1]^X$ such that, for each $\bar{\mu} = (\mu_1, \mu_2) \in \mathbb{I}^X$, the element $S(\bar{\mu}) = S(\mu_1, \mu_2) \in [0,1]^X$ is defined for each $x \in [0,1]$ by $S(\bar{\mu})(x) = S(\mu_1(x), \mu_2(x))$.

However, there is no one way to assign a value of supplementarity to a pair of fuzzy sets. Hence, it is worthwhile proposing an axiomatic definition that establishes the minimum conditions for deciding whether a function is suitable for measuring supplementarity.

Definition 2.3. Given $X \neq \emptyset$ and a t-conorm S, a function $\mathcal{S} : \mathbb{I}^X \to [0,1]$ is said to be a S*-supplementary measure* if it satisfies:

s.1) $\mathcal{S}(\bar{\mu}_X) = \mathcal{S}(\mu_X, \mu_X) = 1$.
s.2) Given $\mu_1, \mu_2 \in [0,1]^X$, if there exists $x \in X$ such that $S(\mu_1(x), \mu_2(x)) < 1$, then $\mathcal{S}(\mu_1, \mu_2) = 0$.
s.3) Symmetry: $\mathcal{S}(\mu_1, \mu_2) = \mathcal{S}(\mu_2, \mu_1)$ for all $\mu_1, \mu_2 \in [0,1]^X$.
s.4) Monotonicity: If $\mu_1, \sigma_1 \in [0,1]^X$ such that $\mu_1 \leq \sigma_1$, then $\mathcal{S}(\mu_1, \mu_2) \leq \mathcal{S}(\sigma_1, \mu_2)$ for all $\mu_2 \in [0,1]^X$.

Remark 2.4. It is easy to see that if $\mathcal{S}$ is a supplementarity measure then $\mathcal{S} \in \mathbb{M}^2([0,1]^X)$ and is a fuzzy $\leq_{\mathbb{I}}$-measure.

2.3. Antonymous measures and supplementarity measures

We can construct supplementarity measures directly; for instance, using geometrical methods as in the case of the incompatibility measures (see [3]). Nevertheless this paper deals with constructions using antonyms, since fuzzy measures are a particular case of fuzzy set, and makes sense to speak of antonyms of fuzzy measures. Hence, we first try to map the concept of antonym to the fuzzy measures framework.

As is well known, given an interval $I \subset \mathbb{R}$ and a strong negation $N : [0,1] \to [0,1]$ [5, 9, 13] (that is, N is decreasing and satisfies $N(0) = 1$, $N(1) = 0$ and $N(N(a)) = a$, $\forall a \in [0,1]$), if $\mu \in [0,1]^I$ is a monotonic function that is the membership function of a fuzzy set $\tilde{A}$, then $\tilde{A}_a$ is an N-antonym of $\tilde{A}$ (or its membership function μ_a is an N-antonym of μ) if the following holds:

1. $\mu_a(x) \leq (N \circ \mu)(x)$ for all $x \in I$.
2. Given $x, y \in I$, then

$$\begin{array}{lcl} \mu(x) < \mu(y) & \Longrightarrow & \mu_a(x) \geq \mu_a(y) \\ \mu_a(x) < \mu_a(y) & \Longrightarrow & \mu(x) \geq \mu(y) \end{array}$$

Since the elements of $\mathbb{M}^2([0,1]^X)$ are $\leq_{\mathbb{I}}$-monotonic, we can map the definition of antonym to $\mathbb{M}^2([0,1]^X)$.

Definition 2.5. Given $X \neq \emptyset$ and a strong negation N, for $M \in \mathbb{M}^2([0,1]^X)$, we say that $M_a \in \mathbb{M}^2([0,1]^X)$ is an N*-antonym* of M if it satisfies:

a.1) $M_a(\bar{\mu}) \leq (N \circ M)(\bar{\mu})$ for all $\bar{\mu} \in \mathbb{M}^2([0,1]^X)$.
a.2) If $\bar{\mu}, \bar{\sigma} \in \mathbb{I}^X$ are $\leq_{\mathbb{I}}$-comparable, then

$$\begin{array}{lcl} M(\bar{\mu}) < M(\bar{\sigma}) & \Longrightarrow & M_a(\bar{\mu}) \geq M_a(\bar{\sigma}) \\ M_a(\bar{\mu}) < M_a(\bar{\sigma}) & \Longrightarrow & M(\bar{\mu}) \geq M(\bar{\sigma}) \end{array}$$

Obviously, if M_1 is an antonymous measure of M_2, then M_2 is an antonymous measure of M_1. Thus, we say that M_1 and M_2 are antonyms.

Let us present a result that relates supplementary measures to other types of measures by antonyms. We refer to incompatibility measures that determine to what extent the intersection of two fuzzy sets is empty. Obviously, the intersection is modeled by a t-norm T (that is, [1, 6, 7, 8] a commutative, associative and monotonic increasing function $T : \mathbb{I} \to [0,1]$ such that $T(a,1) = a$ for all $a \in [0,1]$).

Definition 2.6. ([2]) Given $X \neq \emptyset$ and a t-norm T, a function $\mathcal{I} : \mathbb{I}^X \to [0,1]$ is said to be a T-*incompatibility measure* if it satisfies:

i.1) $\mathcal{I}(\bar{\mu}_\emptyset) = \mathcal{I}(\mu_\emptyset, \mu_\emptyset) = 1$.
i.2) Given $\mu_1, \mu_2 \in [0,1]^X$, if there exists $x \in X$ such that $T(\mu_1(x), \mu_2(x)) > 0$, then $\mathcal{I}(\mu_1, \mu_2) = 0$.
i.3) Symmetry: $\mathcal{I}(\mu_1, \mu_2) = \mathcal{I}(\mu_2, \mu_1)$ for all $\mu_1, \mu_2 \in [0,1]^X$.
i.4) Anti-monotonicity: If $\mu_1, \sigma_1 \in [0,1]^X$ such that $\mu_1 \leq \sigma_1$, then $\mathcal{I}(\mu_1, \mu_2) \geq \mathcal{I}(\sigma_1, \mu_2)$ for all $\mu_2 \in [0,1]^X$.

Remark 2.7. It is easy to see that if $\mathcal{I}$ is an incompatibility measure then $\mathcal{I} \in \mathbb{M}^2([0,1]^X)$ and is a fuzzy $\geq_{\mathbb{I}}$-measure.

Before introducing the result relating supplementarity and incompatibility measures, we need to recall some concepts and results. The Łukasiewicz t-norm, W, and t-conorm, W^*, are defined for each $(a_1, a_2) \in \mathbb{I}$ by $W(a_1, a_2) = \max\{0, a_1 + a_2 - 1\}$ and $W^*(a_1, a_2) = \min\{a + b, 1\}$. If $\varphi \in \mathcal{A}([0,1]) = \{\varphi : [0,1] \to [0,1] \mid \varphi \text{ is an increasing bijection}\}$ (that is, φ is an automorphism of $[0,1]$) and $T, S : \mathbb{I} \to [0,1]$ are, respectively, a t-norm and a t-conorm, then $T_\varphi = \varphi^{-1} \circ T \circ (\varphi \times \varphi)$ and $S_\varphi = \varphi^{-1} \circ S \circ (\varphi \times \varphi)$ are also a t-norm and a t-conorm. In particular, we denote $W^*_\varphi = (W^*)_\varphi = \varphi^{-1} \circ W^* \circ (\varphi \times \varphi)$.

Theorem 2.8. *Let $X \neq \emptyset$ and $\varphi \in \mathcal{A}([0,1])$. If $\mathcal{I}$ is a W_φ-incompatibility measure and $\mathcal{S}$ is a W^*_φ-supplmentary measures such that for all $\bar{\mu} = (\mu_1, \mu_2) \in \mathbb{I}^X$ satisfying $\varphi(\mu_1(x)) + \varphi(\mu_2(x)) = 1$ for all $x \in X$, $\mathcal{I}(\bar{\mu}) = 0$ or $\mathcal{S}(\bar{\mu}) = 0$ holds; then, $\mathcal{I}$ and $\mathcal{S}$ are N-antonyms for any strong negation N.*

Proof. a.1) Let us see that axiom a.1 of antonymous measures (Def.2.5) is satisfied.

First, let us observe that, as $W_\varphi(a_1, a_2) = \varphi^{-1}(\max\{0, \varphi(a_1) + \varphi(a_2) - 1\})$ and $W^*_\varphi(a_1, a_2) = \varphi^{-1}(\min\{\varphi(a_1) + \varphi(a_2), 1\})$, then

$$W_\varphi(a_1, a_2) = 0 \iff \varphi(a_1) + \varphi(a_2) \leq 1 \tag{1}$$

$$W^*_\varphi(a_1, a_2) = 1 \iff \varphi(a_1) + \varphi(a_2) \geq 1 \tag{2}$$

Now, given $\bar{\mu} = (\mu_1, \mu_2) \in \mathbb{I}^X$, we can consider three possibilities:

1. There exists $x \in X$ such that $\varphi(\mu_1(x)) + \varphi(\mu_2(x)) > 1$, then it follows, from (1), that $W_\varphi(\bar{\mu}(x)) > 0$ and, according to axiom i.2 of Definition 2.6, $\mathcal{I}(\bar{\mu}) = 0$; hence $(N \circ \mathcal{I})(\bar{\mu}) = 1 \geq \mathcal{S}(\bar{\mu})$.
2. There exists $x \in X$ such that $\varphi(\mu_1(x)) + \varphi(\mu_2(x)) < 1$, then it follows, from (2), that $W^*_\varphi(\bar{\mu}(x)) < 1$ and, according to axiom s.2 of Definition 2.3, $\mathcal{S}(\bar{\mu}) = 0$; hence $\mathcal{S}(\bar{\mu}) = 0 \leq (N \circ \mathcal{I})(\bar{\mu})$.
3. $\varphi(\mu_1(x)) + \varphi(\mu_2(x)) = 1$ for all $x \in X$, then by hypothesis $\mathcal{I}(\bar{\mu}) = 0$ or $\mathcal{S}(\bar{\mu}) = 0$, and $\mathcal{S}(\bar{\mu}) \leq (N \circ \mathcal{I})(\bar{\mu})$ in both cases.

a.2) Let $\bar{\mu}, \bar{\sigma} \in \mathbb{I}^X$ be $\leq_{\mathbb{I}}$-comparable, then:

$$\text{If } \bar{\mu} \leq_{\mathbb{I}} \bar{\sigma} \Longrightarrow \begin{cases} \mathcal{S}(\bar{\mu}) \leq \mathcal{S}(\bar{\sigma}) \\ \mathcal{I}(\bar{\mu}) \geq \mathcal{I}(\bar{\sigma}) \end{cases}$$

and

$$\text{if } \bar{\sigma} \leq_{\mathbb{I}} \bar{\mu} \Longrightarrow \begin{cases} \mathcal{S}(\bar{\sigma}) \leq \mathcal{S}(\bar{\mu}) \\ \mathcal{I}(\bar{\sigma}) \geq \mathcal{I}(\bar{\mu}) \end{cases}.$$

Thus, axiom a.2 is satisfied.

□

Remark 2.9. Note that the thesis of this theorem will not be true unless $\mathcal{S}(\bar{\mu}) = 0$ or $\mathcal{I}(\bar{\mu}) = 0$ if $\varphi(\mu_1(x)) + \varphi(\mu_2(x)) = 1$ for all $x \in X$. If not, we can consider the following measures: Given $X \neq \emptyset$, $\varphi \in \mathcal{A}([0,1])$ and $\bar{\mu} = (\mu_1, \mu_2) \in \mathbb{I}^X$, define

$$\mathcal{S}(\bar{\mu}) = \begin{cases} 1 & \text{if } \varphi(\mu_1(x)) + \varphi(\mu_2(x)) \geq 1 \ \forall x \in X \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathcal{I}(\bar{\mu}) = \begin{cases} 1 & \text{if } \varphi(\mu_1(x)) + \varphi(\mu_2(x)) \leq 1 \ \forall x \in X \\ 0 & \text{otherwise} \end{cases}.$$

Thus $\mathcal{S}$ is a W^*_φ-supplementarity measure and $\mathcal{I}$ is a W_φ-incompatibility measure. Nevertheless they are not N-antonyms for any strong negation N, since if we consider $\bar{\mu} \in \mathbb{I}^X$ such that $\bar{\mu}(x) = (0,1)$ for all $x \in X$, then

$$1 = \mathcal{S}(\bar{\mu}) \nleq (N \circ \mathcal{I})(\bar{\mu}) = N(1) = 0.$$

3. Constructing supplementarity measures using antonymous

Now that we know what it means for two fuzzy measures to be antonyms, let us consider how to construct antonyms of a measure. The usual way to build antonyms of fuzzy sets defined on real intervals is by means of involutions [11, 12]. Thus, given a monotonic fuzzy set $\mu \in [0,1]^X$, where $X = [a,b] \subset \mathbb{R}$, we can define a decreasing function $\alpha : [a,b] \to [a,b]$ such that $\alpha(a) = b$, $\alpha(b) = a$ and $\alpha(\alpha(x)) = x$ for all $x \in [a,b]$ (that is, α is an involution on $[a,b]$). Often we find a strong negation N such that $\mu \circ \alpha \leq N \circ \mu$, then $\mu_a = \mu \circ \alpha$ satisfies the antonym axioms. In order to map this method to the $\leq_{\mathbb{I}}$-monotonic fuzzy measures framework, we must introduce the concept of involution on the universe $\mathbb{I}^X$. To do this, in the following section, we first define and discuss the involution concept on $\mathbb{I}$, which can be used to define the same concept in $\mathbb{I}^X$.

3.1. Involutions in the unit square

Definition 3.1. A function $\alpha : \mathbb{I} \to \mathbb{I}$ is said to be an *involution* on $\mathbb{I}$ if:

1. α satisfies the following boundary conditions: $\alpha(0_{\mathbb{I}}) = 1_{\mathbb{I}}$ and $\alpha(1_{\mathbb{I}}) = \alpha(0_{\mathbb{I}})$.
2. α is monotonic decreasing: If $\bar{a}, \bar{b} \in \mathbb{I}$ such that $\bar{a} \leq_{\mathbb{I}} \bar{b}$, then $\alpha(\bar{b}) \leq_{\mathbb{I}} \alpha(\bar{a})$.
3. α is involutive: $\alpha(\alpha(\bar{a})) = \bar{a}$ for all $\bar{a} \in \mathbb{I}$.

In what follows, we present a representation theorem for involutions on $\mathbb{I}$ together with the results needed to prove it. From now on, we denote by Π_1 and Π_2 the first and second projections of $\mathbb{R}^2$ into $\mathbb{R}$, that is, for each $(a_1, a_2) \in \mathbb{R}^2$, $\Pi_1(a_1, a_2) = a_1$ and $\Pi_2(a_1, a_2) = a_2$.

We omit the proof of the following lemma as it is based on the same idea as the proof of Lemma 3.5

Lemma 3.2. Let α be an involution on $\mathbb{I}$, then

$$\begin{array}{c} \alpha(0,1) = (0,1) \\ \alpha(1,0) = (1,0) \end{array} \quad \text{or} \quad \begin{array}{c} \alpha(0,1) = (1,0) \\ \alpha(1,0) = (0,1) \end{array}$$

hold.

The following two results show how each segment of the boundary of $\mathbb{I}$ (that we denote by $\partial\,\mathbb{I}$) is mapped by α to another segment. Thus, the boundary is invariant with respect to α, that is, $\alpha(\partial\,\mathbb{I}) = \partial\mathbb{I}$.

Corollary 3.3. Let α be an involution on $\mathbb{I}$, the following holds:

1. If $\alpha(0,1) = (0,1)$ then
$$\begin{array}{c} \Pi_1(\alpha(a_1,1)) = 0 \\ \Pi_1(\alpha(a_1,0)) = 1 \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_2(\alpha(1,a_2)) = 0 \\ \Pi_2(\alpha(0,a_2)) = 1. \end{array}$$
2. If $\alpha(0,1) = (1,0)$ then
$$\begin{array}{c} \Pi_2(\alpha(a_1,1)) = 0 \\ \Pi_2(\alpha(a_1,0)) = 1 \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_1(\alpha(1,a_2)) = 0 \\ \Pi_1(\alpha(0,a_2)) = 1. \end{array}$$

Proof. Let us prove the first part of point 1, the proof of the other equalities runs similarly.

Since $(0,1) \leq_{\mathbb{I}} (a_1,1)$ and α is decreasing, then $\alpha(a_1,1) \leq_{\mathbb{I}} \alpha(0,1) = (0,1)$; hence $\Pi_1(\alpha(a_1,1)) = 0$.

As $(a_1,0) \leq_{\mathbb{I}} (1,0)$, α is decreasing and it follows, from the previous lemma, that $\alpha(1,0) = (1,0)$, then we have that $\alpha(1,0) = (1,0) \leq_{\mathbb{I}} \alpha(a_1,0)$; hence $1 = \Pi_1(\alpha(a_1,0))$. □

Corollary 3.4. Let L_i, with $i = 1 \ldots 4$, be the four closed segments that constitute $\partial\,\mathbb{I}$ where $L_1 = \{(a_1,0) \mid a_1 \in [0,1]\}$ and considering the segments to arranged clockwise; the following holds:

1. If $\alpha(0,1)$=$(0,1)$ then $\alpha|_{L_1}$ is a bijection from L_1 onto L_4 and $\alpha|_{L_2}$ is a bijection from L_2 onto L_3.
2. If $\alpha(0,1)$=$(1,0)$ then $\alpha|_{L_1}$ is a bijection from L_1 onto L_3 and $\alpha|_{L_2}$ is a bijection from L_2 onto L_4.

Proof. Let us prove the first part of point 1, the proof of the other statements runs similarly.

First, observe that α is injective in the whole $\mathbb{I}$; indeed, if $\alpha(\bar{a}) = \alpha(\bar{b})$, it follows that $\bar{a} = \bar{b}$ as α is involutive. According to Corollary 3.3, $\alpha(L_1) \subset L_4 = \{(1,a_2) \mid a_2 \in [0,1]\}$. Moreover, for each $(1,a_2) \in L_4$, it follows, again from Corollary 3.3, that $\bar{b} = \alpha(1,a_1) \in L_1$ and, as α is involutive, then $\alpha(\bar{b}) = (1,a_1)$; hence $\alpha|_{L_1}$ is surjective. □

Lemma 3.5. Let α be an involution on $\mathbb{I}$, the following holds:

1. If $\alpha(0,1) = (0,1)$, then
$$\begin{array}{ll} \Pi_1(\alpha(a_1,a_2)) = \Pi_1(\alpha(0,a_2)), & \forall\, a_1 \in [0,1] \\ \Pi_2(\alpha(a_1,a_2)) = \Pi_2(\alpha(a_1,0)), & \forall\, a_2 \in [0,1]. \end{array}$$
2. If $\alpha(0,1) = (1,0)$, then
$$\begin{array}{ll} \Pi_1(\alpha(a_1,a_2)) = \Pi_1(\alpha(a_1,0)), & \forall\, a_2 \in [0,1] \\ \Pi_2(\alpha(a_1,a_2)) = \Pi_2(\alpha(0,a_2)), & \forall\, a_1 \in [0,1]. \end{array}$$

Proof. Let us prove the first equality of the second point, similar arguments apply for the others. We have that

$$(a_1,0) \leq_{\mathbb{I}} (a_1,a_2) \leq_{\mathbb{I}} (a_1,1), \;\; \forall\, a_2 \in [0,1],$$

then, as α is decreasing,

$$\alpha(a_1,1) \leq_{\mathbb{I}} \alpha(a_1,a_2) \leq_{\mathbb{I}} \alpha(a_1,0), \;\; \forall\, a_2 \in [0,1]$$

holds, and so, for all $a_2 \in [0,1]$,

$$\Pi_1(\alpha(a_1,1)) \leq \Pi_1(\alpha(a_1,a_2)) \leq \Pi_1(\alpha(a_1,0))$$

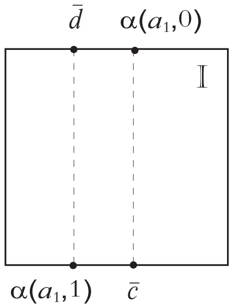
Now, let us see that $\Pi_1(\alpha(a_1,1)) = \Pi_1(\alpha(a_1,0))$, which will prove the expected equality. Suppose that $\Pi_1(\alpha(a_1,1)) \lneqq \Pi_1(\alpha(a_1,0))$, thus we can consider the elements $\bar{c} = (\Pi_1(\alpha(a_1,0),0)$ and $\bar{d} = (\Pi_1(\alpha(a_1,1)),1)$ (see the figure) that are non $\leq_{\mathbb{I}}$-comparable. Moreover, as it follows, from Corollary 3.3, that $\alpha(a_1,1) = (\Pi_1(\alpha(a_1,1)),0)$ and $\alpha(a_1,0) = (\Pi_1(\alpha(a_1,0)),1)$, $\bar{c}$ and $\bar{d}$ satisfy

$$\alpha(a_1,1) \leq_{\mathbb{I}} \bar{c} \leq_{\mathbb{I}} \alpha(a_1,0)$$
$$\alpha(a_1,1) \leq_{\mathbb{I}} \bar{d} \leq_{\mathbb{I}} \alpha(a_1,0).$$

Then, since α is decreasing and involutive, we obtain

$$(a_1,0) \leq_{\mathbb{I}} \alpha(\bar{c}) \leq_{\mathbb{I}} (a_1,1)$$
$$(a_1,0) \leq_{\mathbb{I}} \alpha(\bar{d}) \leq_{\mathbb{I}} (a_1,1),$$

and thus, the elements $\alpha(\bar{c})$ and $\alpha(\bar{d})$ are $\leq_{\mathbb{I}}$-comparable since the order $\leq_{\mathbb{I}}$ is linear on the segment that joins $(a_1,0)$ to $(a_1,1)$. Hence, $\alpha(\alpha(\bar{c})) = \bar{c}$ and $\alpha(\alpha(\bar{d})) = \bar{d}$ are $\leq_{\mathbb{I}}$-comparable, which is false, and we can conclude that $\Pi_1(\alpha(a_1,1)) = \Pi_1(\alpha(a_1,0))$. □



Theorem 3.6. (Representation of involutions) *Let α be an involution on $\mathbb{I}$, then either there exist two strong negations N_1 and N_2 such that $\alpha(a_1, a_2) = (N_1(a_1), N_2(a_2))$, for each $(a_1, a_2) \in \mathbb{I}$, or there exists a negation N such that $\alpha(a_1, a_2) = (N(a_2), N^{-1}(a_1))$, for each $(a_1, a_2) \in \mathbb{I}$.*

Proof. First, suppose that $\alpha(0, 1) = (1, 0)$. Define $N_1(a_1) = \Pi_1(\alpha(a_1, 0))$ for each $a_1 \in [0, 1]$ and $N_2(a_2) = \Pi_2(\alpha(0, a_2))$ for each $a_2 \in [0, 1]$. Then:

1. N_1 and N_2 are negations. i) $N_1(0) = \Pi_1(\alpha(0, 0)) = \Pi_1(1, 1) = 1$. ii) From Lemma 3.2, $\alpha(1, 0) = (0, 1)$, then $N_1(1) = \Pi_1(\alpha(1, 0)) = \Pi_1(0, 1) = 0$. iii) If $a_1, b_1 \in [0, 1]$ such that $a_1 \leq b_1$, then $(a_1, 0) \leq_{\mathbb{I}} (b_1, 0)$ and, as α is decreasing, $\alpha(b_1, 0) \leq_{\mathbb{I}} \alpha(a_1, 0)$, from which we can conclude that $N(b_1) \leq N(a_1)$. Analogously, for N_2.
2. For each $(a_1, a_2) \in \mathbb{I}$, it follows, from Lemma 3.5, that
$$\begin{aligned}\alpha(a_1, a_2) &= (\Pi_1(\alpha(a_1, a_2)), \Pi_2(\alpha(a_1, a_2)))\\ &= (\Pi_1(\alpha(a_1, 0)), \Pi_2(\alpha(0, a_2)))\\ &= (N_1(a_1), N_2(a_2)).\end{aligned}$$
3. Finally, N_1 and N_2 are involutive. Indeed, as α is involutive, then
$$\begin{aligned}(a_1, a_2) &= \alpha(\alpha(a_1, a_2)) = \alpha(N_1(a_1), N_2(a_2))\\ &= (N_1(N_1(a_1)), N_2(N_2(a_2))).\end{aligned}$$

Second, suppose that $\alpha(0, 1) = (0, 1)$. Define $N(a_2) = \Pi_1(\alpha(0, a_2))$ for each $a_2 \in [0, 1]$ and $N^*(a_1) = \Pi_2(\alpha(a_1, 0))$ for each $a_1 \in [0, 1]$. Then:

1. As in the first case, we find that N and N^* are negations.
2. For each $(a_1, a_2) \in \mathbb{I}$, it follows,from Lemma 3.5, that
$$\begin{aligned}\alpha(a_1, a_2) &= (\Pi_1(\alpha(a_1, a_2)), \Pi_2(\alpha(a_1, a_2)))\\ &= (\Pi_1(\alpha(0, a_2)), \Pi_2(\alpha(a_1, 0)))\\ &= (N(a_2), N^*(a_1)).\end{aligned}$$
3. $N^* = N^{-1}$. Indeed, for each $(a_1, a_2) \in \mathbb{I}$, it holds that
$$\begin{aligned}(a_1, a_2) &= \alpha(\alpha(a_1, a_2)) = \alpha(N(a_2), N^*(a_1))\\ &= (N(N^*(a_1)), N^*(N(a_2))).\end{aligned}$$
Thus $N(N^*(a)) = a = N^*(N(a))$ for all $a \in [0, 1]$, that is, $N^{-1} = N^*$.

$\square$

3.2. Supplementarity measures as antonymous measures

In order to construct antonymous measures, we extend the concept of involution on $\mathbb{I}$ to $\mathbb{I}^X$.

Definition 3.7. A function $\alpha : \mathbb{I}^X \to \mathbb{I}^X$ is said to be an *involution* on $\mathbb{I}^X$ if:

1. α satisfies the following boundary conditions: $\alpha(\bar{\mu}_\emptyset) = \bar{\mu}_X$ and $\alpha(\bar{\mu}_X) = \alpha(\bar{\mu}_\emptyset)$.
2. α is monotonic decreasing: If $\bar{\mu}, \bar{\sigma} \in \mathbb{I}^X$ such that $\bar{\mu} \leq_{\mathbb{I}} \bar{\sigma}$, then $\alpha(\bar{\sigma}) \leq_{\mathbb{I}} \alpha(\bar{\mu})$.
3. α is involutive: $\alpha(\alpha(\bar{\mu})) = \bar{\mu}$ for all $\bar{\mu} \in \mathbb{I}^X$.

It is easy to prove the following result that can be used to construct involutions on $\mathbb{I}^X$ based on involutions on $\mathbb{I}$.

Proposition 3.8. Given an involution α on $\mathbb{I}$ and a universal set $X \neq \emptyset$, the function $\widehat{\alpha} : \mathbb{I}^X \to \mathbb{I}^X$ defined for each $\bar{\mu} \in \mathbb{I}^X$ by $\widehat{\alpha}(\bar{\mu}) = \alpha \circ \bar{\mu}$ is an involution on $\mathbb{I}^X$.

Example 3.9. Let $X \neq \emptyset$ and N be a strong negation, consider the functions $\widehat{\alpha}_1, \widehat{\alpha}_2 : \mathbb{I}^X \to \mathbb{I}^X$ defined for each $\bar{\mu} = (\mu_1, \mu_2) \in \mathbb{I}^X$ by

$$\begin{aligned}\widehat{\alpha}_1(\bar{\mu}) &= (N \circ \mu_1, N \circ \mu_2)\\ \widehat{\alpha}_2(\bar{\mu}) &= (N \circ \mu_2, N^{-1} \circ \mu_1).\end{aligned}$$

Then, from Theorem 3.6 of the representation of involutions on $\mathbb{I}$ and the Proposition 3.8, it follows that $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ are involutions on $\mathbb{I}^X$.

Remember that the function $N_\varphi : [0, 1] \to [0, 1]$ defined for each $a \in [0, 1]$ by $N_\varphi(a) = \varphi^{-1}(1 - \varphi(a))$ is a strong negation; moreover the strong negations in [9] are characterized by this formula.

Theorem 3.10. *Let $X \neq \emptyset$ and $\varphi \in \mathcal{A}([0, 1])$. Consider the strong negation N_φ and the involutions $\widehat{\alpha}_1, \widehat{\alpha}_2 : \mathbb{I}^X \to \mathbb{I}^X$ defined, respectively, for each $\bar{\mu} = (\mu_1, \mu_2)$, by*

$$\widehat{\alpha}_1(\bar{\mu}) = (N_\varphi \circ \mu_1, N_\varphi \circ \mu_2)$$

$$\widehat{\alpha}_2(\bar{\mu}) = (N_\varphi \circ \mu_2, N_\varphi \circ \mu_1).$$

If $\mathcal{I} \in \mathbb{M}^2([0, 1]^X)$ is a W_φ-incompatibility measure, then $\mathcal{S}_1 = \mathcal{I} \circ \widehat{\alpha}_1, \mathcal{S}_2 = \mathcal{I} \circ \widehat{\alpha}_2 \in \mathbb{M}^2([0, 1]^X)$ are W^*_φ-supplementarity measures.

Proof. We will give the proof for $\mathcal{S}_1$ only. The proof for $\mathcal{S}_2$ is similar. Axioms s.1, s.3 and s.4 follow straightforwardly. Let us look at axiom s.2. Let $\bar{\mu} = (\mu_1, \mu_2) \in \mathbb{I}^X$ and suppose that there exists $x \in X$ such that $W^*_\varphi(\bar{\mu}(x)) < 1$. Then it follows, from equation (2) in Theorem 2.8, that $\varphi(\mu_1(x)) + \varphi(\mu_2(x)) < 1$, which is equivalent to

$$1 - \varphi(\mu_1(x)) + 1 - \varphi(\mu_2(x)) > 1$$

and this is equivalent to

$$\varphi(\varphi^{-1}(1 - \varphi(\mu_1(x)))) + \varphi(\varphi^{-1}(1 - \varphi(\mu_2(x)))) > 1,$$

that is,

$$\varphi((N_\varphi \circ \mu_1)(x)) + \varphi((N_\varphi \circ \mu_2)(x)) > 1.$$

Then it follows, from equation (1) in Theorem 2.8, that

$$W_\varphi((N_\varphi \circ \mu_1)(x), (N_\varphi \circ \mu_2)(x)) > 0.$$

Hence, from axiom i.2 of incompatibility measures, we have

$$\mathcal{S}_1(\bar{\mu}) = (\mathcal{I} \circ \widehat{\alpha}_1)(\bar{\mu}) = \mathcal{I}(N_\varphi \circ \mu_1, N_\varphi \circ \mu_2) = 0.$$

$\square$

Example 3.11. Given $X \neq \emptyset$ and $\varphi \in \mathcal{A}([0,1])$, consider the W_φ-incompatibility measure [2] $\mathcal{I}_\varphi \in \mathbb{M}^2([0,1]^X)$ defined for each $\bar{\mu} = (\mu_1, \mu_2) \in \mathbb{I}^X$ by

$$\mathcal{I}_\varphi(\bar{\mu}) = \max\left\{0, 1 - \sup_{x \in X}(\varphi(\mu_1(x)) + \varphi(\mu_2(x)))\right\}.$$

Then the function $\mathcal{S}_\varphi \in \mathbb{M}^2([0,1]^X)$ defined for each $\bar{\mu} = (\mu_1, \mu_2) \in \mathbb{I}^X$ by

$$\begin{aligned}\mathcal{S}_\varphi(\bar{\mu}) &= \mathcal{I}_\varphi(N_\varphi \circ \mu_1, N_\varphi \circ \mu_2)\\ &= \max\left\{0, \inf_{x \in X}(\varphi(\mu_1(x)) + \varphi(\mu_2(x))) - 1\right\}\end{aligned}$$

is a W^*_φ-supplementarity measure.

Remark 3.12. Theorem 3.10 provides a mechanism for constructing W^*_φ-supplementarity measures from W_φ-incompatibility measures by means of involutions. These measures, moreover, are antonymous, according to Theorem 2.8.

Conclusions

In this paper we report the early stages of a study of supplementarity within the fuzzy sets framework, where we present an axiomatic definition setting out the minimum requirements that a function should satisfy for use as such a measure.

After having defined the supplementarity measures (with respect to a t-conorm S) between two fuzzy sets, we have defined N-antonymous measures for a strong negation N. To be able to find a mechanism that we can use to find antonymous measures of another measure, we have introduced the concept of involution on $[0,1] \times [0,1]$, providing an involution representation theorem. We then extended this concept to $\mathbb{I}^X = [0,1]^X \times [0,1]^X$

We have also shown how each W^*_φ-supplementarity measure is antonymous with each W^*-incompatibility measure and have also built W^*_φ-supplementarity measures as antonyms of W_φ-incompatibility using involutions.

Some of the immediate and future lines of research in continuation of this work are: an investigation of the continuity of the supplementarity measures proposed here, use of the proposed measures in applications to evaluate the results and extension of the study of supplementarity to other fuzzy logic structures, like, for example, the field of Atanassov's intuitionistic fuzzy sets.

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